FUNCTION CONTRACTIVE MAPS IN TRIANGULAR SYMMETRIC SPACES

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Abstract. Some fixed point results are given for a class of functional contractions acting on (reflexive) triangular symmetric spaces. Technical connections with the corresponding theories over (standard) metric and partial metric spaces are also being established.

1. Introduction

Let X be a nonempty set. By a *symmetric* over X we shall mean any map $d : X \times X \to R_+ := [0, \infty[$ with (cf. Hicks [9])

(a01) $d(x, y) = d(y, x)$, $\forall x, y \in X$;

the couple (X, d) will be referred to as a *symmetric space*. Call d, *triangular*, if

(a02) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$;

and *reflexive-triangular*, provided it fulfills (the stronger condition)

(a03) $d(x, z) + d(y, y) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

The class of such particular spaces has multiple connections with the one of *partial metric spaces*, due to Matthews [14]. For, as we shall see below, the fixed point theory for functional contractive maps in (reflexive) triangular symmetric spaces is a common root of both corresponding theories in standard metric spaces and partial metric spaces. This ultimately tells us that, for most of the function contractions taken from the list in Rhoades [16], any such theory over partial metric spaces is nothing but a clone of the corresponding one developed for standard metric spaces. Further aspects will be delineated elsewhere.

2. Preliminaries

Let (X, d) be a symmetric space; where $d(., .)$ is triangular. Call the subset Y of X, *d-singleton* provided $y_1, y_2 \in Y \Longrightarrow d(y_1, y_2) = 0$.

(A) We introduce a $0d$-convergence and $0d$-Cauchy structure on X as follows. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), $0d$-converges to x (written as: $x_n \xrightarrow{0d} x$) provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

(b01) $\forall \varepsilon > 0$, $\exists i = i(\varepsilon)$: $n \geq i \Longrightarrow d(x_n, x) < \varepsilon$.

The set of all such points x will be denoted $0d - \lim_n(x_n)$; note that, it is a d-singleton, because d is triangular. If $0d - \lim_n(x_n)$ is nonempty, then (x_n) is called *0d-convergent*. We stress that the concept (b01) does not match the standard requirements in Kasahara [13]; because, for the constant sequence $(x_n = u; n \geq 0)$,

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we do not have $x_n \xrightarrow{0d} u$ if $d(u,u) \neq 0$. Further, call the sequence (x_n), $0d$-*Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m \neq n$; i.e.,

(b02) $\forall \varepsilon > 0$, $\exists j = j(\varepsilon)$: $j \le m < n \Longrightarrow d(x_m, x_n) < \varepsilon$.

As d is triangular, any $0d$-convergent sequence is $0d$-Cauchy too; but, the reciprocal is not in general true. Let us say that (X, d) is *0-complete*, if each $0d$-Cauchy sequence is $0d$-convergent.

(B) Call the sequence $(x_n; n \ge 0)$, $0d$-*semi-Cauchy* provided

(b03) $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$.

Clearly, each $0d$-Cauchy sequence is $0d$-semi-Cauchy; but not conversely. The following auxiliary statement about such objects is useful for us.

Lemma 1. *Let $(x_n; n \ge 0)$ be a $0d$-semi-Cauchy sequence in X that is not $0d$-Cauchy. There exist then $\varepsilon > 0$, $j(\varepsilon) \in N$, and a couple of rank-sequences $(m(j); j \ge 0)$, $(n(j); j \ge 0)$ with*

$$j \le m(j) < n(j), \quad d(x_{m(j)}, x_{n(j)}) \ge \varepsilon, \quad \forall j \ge 0 \tag{2.1}$$

$$n(j) - m(j) \ge 2, \quad d(x_{m(j)}, x_{n(j)-1}) < \varepsilon, \ \forall j \ge j(\varepsilon) \tag{2.2}$$

$$\lim_j d(x_{m(j)+p}, x_{n(j)+q}) = \varepsilon, \quad \forall p, q \in \{0, 1\}. \tag{2.3}$$

Proof. As $(x_n; n \ge 0)$ is not $0d$-Cauchy, there exists, via (b02), an $\varepsilon > 0$ with

$$A(j) := \{(m,n) \in N \times N; j \le m < n, d(x_m, x_n) \ge \varepsilon\} \neq \emptyset, \ \forall j \ge 0.$$

Having this precise, denote, for each $j \ge 0$,

$$m(j) = \min \mathrm{Dom}(A(j)), \ n(j) = \min A(m(j)).$$

As a consequence, the couple of rank-sequences $(m(j); j \ge 0)$, $(n(j); j \ge 0)$ fulfills (2.1). On the other hand, letting the index $j(\varepsilon) \ge 0$ be such that

$$d(x_k, x_{k+1}) < \varepsilon, \quad \forall k \ge j(\varepsilon), \tag{2.4}$$

it is clear that (2.2) holds too. Finally, by the triangular property,

$$\begin{aligned} &\varepsilon \le d(x_{m(j)}, x_{n(j)}) \le d(x_{m(j)}, x_{n(j)-1}) + d(x_{n(j)-1}, x_{n(j)}) \\ &< \varepsilon + d(x_{n(j)-1}, x_{n(j)}), \quad \forall j \ge j(\varepsilon); \end{aligned}$$

and this establishes the case $(p = 0, q = 0)$ of (2.3). Combining with

$$\begin{aligned} &d(x_{m(j)}, x_{n(j)}) - d(x_{n(j)}, x_{n(j)+1}) \le d(x_{m(j)}, x_{n(j)+1}) \\ &\le d(x_{m(j)}, x_{n(j)}) + d(x_{n(j)}, x_{n(j)+1}), \quad \forall j \ge j(\varepsilon) \end{aligned}$$

yields the case $(p = 0, q = 1)$ of the same. The remaining situations are deductible in a similar way. $\square$

Note finally, that the exposed facts do not exhaust the whole completeness theory of such objects applicable to partial metric spaces. Some complementary aspects involving these last objects may be found in Oltra and Valero [15].

(C) Let $\mathcal{F}(A)$ stand for the class of all functions from A to itself. For any $\varphi \in \mathcal{F}(R_+)$, and any s in $R_+^0 :=]0, \infty[$, put

(b04) $L_+\varphi(s) = \inf_{\varepsilon > 0} \Phi[s+](\varepsilon)$; where $\Phi[s+](\varepsilon) = \sup\{\varphi(t); s \le t < s + \varepsilon\}$.

By this very definition, we have the representation

$$L_+\varphi(s) = \max\{\limsup_{t\to s+} \varphi(t), \varphi(s)\}, \quad \forall s \in R_+^0. \tag{2.5}$$

Clearly, the quantity in the right member may be infinite. A basic situation when this cannot hold may be described as follows. Call $\varphi \in \mathcal{F}(R_+)$, *normal* when $[\varphi(0) = 0;\ \varphi(t) < t,\ \forall t > 0]$. Note that, under such a property, one has

$$\varphi(s) \le L_+\varphi(s) \le s, \quad \forall s \in R_+^0. \tag{2.6}$$

The following consequence of this will be useful. Given the sequence $(t_n; n \ge 0)$ in R_+ and $s \in R_+$, define $t_n \downarrow s$ (as $n \to \infty$) provided $[t_n \ge s, \forall n]$ and $t_n \to s$.

Lemma 2. *Let the function $\varphi \in \mathcal{F}(R_+)$ be normal; and $s \in R_+^0$ be arbitrary fixed. Then,*

i) $\limsup_n \varphi(t_n) \le L_+\varphi(s)$*, for each sequence (t_n) with $t_n \downarrow s$*

ii) *there exists a sequence (r_n) with $r_n \downarrow s$ and $\varphi(r_n) \to L_+\varphi(s)$.*

Proof. i) Given $\varepsilon > 0$, there exists a rank $p(\varepsilon) \ge 0$ such that $s \le t_n < s + \varepsilon$, for all $n \ge p(\varepsilon)$; hence

$$\limsup_n \varphi(t_n) \le \sup\{\varphi(t_n); n \ge p(\varepsilon)\} \le \Phi[s+](\varepsilon).$$

It suffices taking the infimum over $\varepsilon > 0$ in this relation to get the desired fact.

ii) From (b04), $L_+\varphi(s) = \inf_{\varepsilon>0} \Phi[s+](\varepsilon)$; so, by the definition of infimum,

$$\forall \varepsilon > 0,\ \exists \delta \in]0, \varepsilon[:\ L_+\varphi(s) \le \Phi[s+](\delta) < L_+\varphi(s) + \varepsilon.$$

This, in the case of $L_+\varphi(s) = 0$, gives the written conclusion with $(r_n = s; n \ge 0)$; for, as a direct consequence of (2.6), one has $\varphi(s) = 0$. Suppose now that $L_+\varphi(s) > 0$. Again from (b04),

$$\forall \varepsilon \in]0, L_+\varphi(s)[,\ \exists \delta \in]0, \varepsilon[:\ L_+\varphi(s) - \varepsilon < L_+\varphi(s) \le \Phi[s+](\delta) < L_+\varphi(s) + \varepsilon.$$

This, along with the definition of supremum, tells us that there must be some r in $[s, s+\delta[$ with $L_+\varphi(s) - \varepsilon < \varphi(r) < L_+\varphi(s) + \varepsilon$. Taking a sequence (ε_n) in R_+^0 with $\varepsilon_n \to 0$, there exists a corresponding sequence (r_n) in R_+ with $r_n \downarrow s$ and $\varphi(r_n) \to L_+\varphi(s)$. Hence the conclusion. □

Call the normal $\varphi \in \mathcal{F}(R_+)$, *right limit normal*, if

(b05) $L_+\varphi(s) < s$ (or, equivalently: $\limsup_{t\to s+} \varphi(t) < s$), $\forall s \in R_+^0$.

[The last assertion is a consequence of (2.5) above]. In particular, the normal function $\varphi \in \mathcal{F}(R_+)$ is right limit normal, whenever it is usc at the right on R_+^0:

(b06) $\limsup_{t\to s+} \varphi(t) \le \varphi(s)$, for each $s \in R_+^0$.

Note that this property is fulfilled when φ is continuous at the right on R_+^0; for, in such a case, $\limsup_{t\to s+} \varphi(t) = \varphi(s)$, $\forall s \in R_+^0$. Another interesting example is furnished by Lemma 2. Let us say that the normal function $\varphi \in \mathcal{F}(R_+)$ is *Geraghty normal* provided (cf. Geraghty [7])

(b07) $(t_n; n \ge 0) \subseteq R_+^0$, $\varphi(t_n)/t_n \to 1$ imply $t_n \to 0$.

Lemma 3. *Let the normal function $\varphi \in \mathcal{F}(R_+)$ be Geraghty normal. Then, φ is necessarily right limit normal.*

Proof. Suppose that the normal function φ is not right limit normal. From (2.6), there exists some $s \in R_+^0$ with $L_+\varphi(s) = s$. Combining with Lemma 2, there exists a sequence $(r_n; n \geq 0)$ with $r_n \downarrow s$ and $\varphi(r_n) \to s$; whence $\varphi(r_n)/r_n \to 1$; i.e.: φ is not Geraghty normal. □

Remark 1. The reciprocal of this is not in general true. In fact, for the (continuous) right limit normal function $[\varphi(t) = t(1 - e^{-t}),\ t \geq 0]$ and the sequence $(t_n = n + 1; n \geq 0)$ in R_+^0, we have $\varphi(t_n)/t_n \to 1$; but, evidently, $t_n \to \infty$.

3. Main result

Let (X, d) be a symmetric space; with, in addition,

(c01) d is triangular and (X, d) is 0-complete.

Further, let $T : X \to X$ be a selfmap of X. Call $z \in X$, *d-fixed* iff $d(z, Tz) = 0$; the class of all such elements will be denoted as $\mathrm{Fix}(T; d)$. Technically speaking, the points in question are obtained by a limit process as follows. Let us say that $x \in X$ is a *Picard point* (modulo (d, T)) if **i)** $(T^n x; n \geq 0)$ is $0d$-convergent, **ii)** each point of $0d - \lim_n T^n x$ is in $\mathrm{Fix}(T; d)$. If this happens for each $x \in X$ then T is referred to as a *Picard operator* (modulo d); if (in addition) $\mathrm{Fix}(T; d)$ is d-singleton, then T is called a *global Picard operator* (modulo d); cf. Rus [18, Ch 2, Sect 2.2].

Now, concrete circumstances guaranteeing such properties involve (in addition to (c01)) contractive selfmaps T with the d-asymptotic property:

(c02) $\lim_n d(T^n x, T^{n+1} x) = 0,\ \forall x \in X$.

Precisely, denote for $x, y \in X$:

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$
$$K(x, y) = (1/2)[d(x, Ty) + d(Tx, y)],\ P(x, y) = \max\{M(x, y), K(x, y)\};$$

and fix $G \in \{M, P\}$. Given $\varphi \in \mathcal{F}(R_+)$, we say that T is $(d, G; \varphi)$-*contractive*, if

(c03) $d(Tx, Ty) \leq \varphi(G(x, y)),\ \forall x, y \in X$.

The main result of this note is

Theorem 1. *Suppose (under (c02)) that T is $(d, G; \varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, T is a global Picard operator (modulo d).*

Proof. Let $z_1, z_2 \in \mathrm{Fix}(T; d)$ be arbitrary fixed. By this very choice,

$$M(z_1, z_2) = \max\{d(z_1, z_2), 0, 0\} = d(z_1, z_2).$$

In addition (from the triangular property)

$$d(z_1, Tz_2) \leq d(z_1, z_2) + d(z_2, Tz_2) = d(z_1, z_2),$$
$$d(z_2, Tz_1) \leq d(z_2, z_1) + d(z_1, Tz_1) = d(z_1, z_2);$$

so that, $K(z_1, z_2) \leq d(z_1, z_2)$; which tells us that $P(z_1, z_2) = d(z_1, z_2)$. On the other hand, again from the choice of our data and the triangular property,

$$d(z_1, z_2) \leq d(z_1, Tz_1) + d(z_2, Tz_2) + d(Tz_1, Tz_2) = d(Tz_1, Tz_2).$$

Combining with the contractive condition yields (for either choice of G)

$$d(z_1, z_2) \leq \varphi(d(z_1, z_2));$$

wherefrom $d(z_1, z_2) = 0$; so that, $\mathrm{Fix}(T; d)$ is d-singleton. It remains now to establish the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \geq 0$; note that, by (c02), $(x_n; n \geq 0)$ is $0d$-semi-Cauchy.

I) We claim that $(x_n; n \geq 0)$ is $0d$-Cauchy. Suppose this is not true. By Lemma 1, there exist $\varepsilon > 0$, $j(\varepsilon) \in N$, and a couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, with the properties (2.1)-(2.3). For simplicity, we shall write (for $j \geq 0$), m, n in place of $m(j), n(j)$ respectively. By the contractive condition,

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + \varphi(G(x_m, x_n)), \quad \forall j \geq j(\varepsilon). \tag{3.1}$$

Denote $(r_j := M(x_m, x_n),\ s_j := K(x_m, x_n),\ t_j := P(x_m, x_n);\ j \geq 0)$. From (2.1), $t_j \geq r_j \geq \varepsilon, \forall j \geq 0$; moreover, (2.3) yields $r_j, s_j, t_j \to \varepsilon$ as $j \to \infty$. So, passing to limit as $j \to \infty$ in (3.1) one gets (via Lemma 2) $\varepsilon \leq L_+\varphi(\varepsilon) < \varepsilon$, contradiction; so that, our assertion follows.

II) As (X, d) is 0-complete, this yields $x_n \xrightarrow{0d} z$ as $n \to \infty$, for some $z \in X$. We claim that z is an element of $\mathrm{Fix}(T; d)$. Suppose not: i.e., $\rho := d(z, Tz) > 0$. By the above properties of $(x_n; n \geq 0)$, there exists $k(\rho) \in N$ such that, for all $n \geq k(\rho)$,

$$d(x_n, x_{n+1}), d(x_n, z) < \rho/2;\ d(x_n, Tz) \leq d(x_n, z) + \rho < 3\rho/2.$$

This gives (again for all $n \geq k(\rho)$)

$$M(x_n, z) = \rho,\ K(x_n, z) < \rho; \quad \text{hence} \quad P(x_n, z) = \rho.$$

By the contractive condition, we then have (for either choice of G)

$$\rho \leq d(z, x_{n+1}) + \varphi(G(x_n, z)) = d(z, x_{n+1}) + \varphi(\rho), \quad \forall n \geq k(\rho).$$

Passing to limit as $n \to \infty$ yields $\rho \leq \varphi(\rho)$, contradiction. Hence, z is an element of $\mathrm{Fix}(T; d)$; and the proof is complete. □

4. Reflexive-triangular case

Now, it remains to determine concrete circumstances under which T is d-asymptotic. Let (X, d) be a symmetric space, with

(d01) d is reflexive-triangular and (X, d) is 0-complete.

Further, let T be a selfmap of X; and fix $G \in \{M, P\}$.

Lemma 4. *Suppose that T is $(d, G; \varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, T is d-asymptotic.*

Proof. By definition, we have

$$M(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x)\}, \quad x \in X. \tag{4.1}$$

On the other hand, by the reflexive-triangular property,

$$K(x, Tx) \leq (1/2)[d(x, Tx) + d(Tx, T^2x)] \leq \max\{d(x, Tx), d(Tx, T^2x)\}.$$

So, by simply combining these,

$$P(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x)\}, \quad x \in X. \tag{4.2}$$

Fix some $x \in X$; and put $(\rho_n := d(T^n x, T^{n+1}x); n \geq 0)$. From the contractive condition, $[\rho_{n+1} \leq \varphi(\max\{\rho_n, \rho_{n+1}\}), \forall n \geq 0]$. As φ is normal, this yields

$$\rho_{n+1} \leq \varphi(\rho_n), \quad \forall n \geq 0.$$

In particular, $(\rho_n; n \geq 0)$ is descending; hence, $\rho := \lim_n \rho_n$ exists in R_+. Assume that $\rho > 0$. By Lemma 2, we must have $\rho \leq L_+\varphi(\rho) < \rho$; contradiction. Hence, $\rho = 0$; and the conclusion follows. □

Now, by simply combining this with Theorem 1, we have (under (d01))

Theorem 2. *Suppose that T is $(d,G;\varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, T is a global Picard operator (modulo d).*

A basic particular case of this result is the following. Call the symmetric d on X, an *almost partial metric* provided it is reflexive-triangular and

(d02) $d(x,y)=0 \Longrightarrow x=y$ (d is *sufficient*).

Note that, in such a case,

$$[\forall Y \in \mathcal{P}_0(X)];\ \ Y \text{ is } d\text{-singleton} \Longrightarrow Y \text{ is singleton} \tag{4.3}$$

$$\mathrm{Fix}(T;d) \subseteq \mathrm{Fix}(T) \text{ (=the class of all fixed points of } T \text{ in } X). \tag{4.4}$$

Assume in the following that

(d03) d is almost partial metric and (X,d) is 0-complete.

Theorem 3. *Let the selfmap T be $(d,G;\varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then,*

$$\mathrm{Fix}(T;d) = \mathrm{Fix}(T) = \{z\}, \ \ \textit{where } d(z,z)=0, \tag{4.5}$$

$$T^n x \xrightarrow{0d} z \ \ \textit{as } n \to \infty, \ \ \textit{for each } x \in X. \tag{4.6}$$

Proof. By Theorem 2, we have (taking (4.3+(4.4) into account)

$$\mathrm{Fix}(T;d) = \{z\}, \text{ with } z \in \mathrm{Fix}(T),\ d(z,z)=0;$$

and, moreover, (4.6) holds. It remains to establish that $\mathrm{Fix}(T)=\{z\}$. For each $w \in \mathrm{Fix}(T)$, we must have (by (4.6)) $T^n w \xrightarrow{0d} z$; which means: $d(w,z)=0$; hence (as d=sufficient) $w=z$. The proof is complete. □

Now, let us give two important examples of such objects.

(A) Clearly, each *(standard) metric* on X is an almost partial metric. Then, Theorem 3 is just the main result in Jachymski [11]. In fact, its argument mimics the one in that paper. The only "specific" fact to be underlined is related to the reflexive-triangular property of our symmetric d.

(B) According to Matthews [14], call the symmetric d, a *partial metric* provided it is reflexive-triangular and

(d04) $[d(x,x)=d(y,y)=d(x,y)] \Longrightarrow x=y$ (d is *strongly sufficient*)

(d05) $\max\{d(x,x),d(y,y)\} \le d(x,y),\ \ \forall x,y \in X$ (*Matthews property*).

Note that, by the reflexive-triangular property, one has (with $z=x$)

$$d(x,x)+d(y,y) \le 2d(x,y), \ \ \forall x,y \in X; \tag{4.7}$$

and this, along with (d04), yields (d02); i.e.: each partial metric is an almost partial metric. Hence, Theorem 3 is applicable to such objects; its corresponding form is just the main result in Romaguera [17]; see also Altun et al [4]. It is to be stressed here that the Matthews property (d05) was not effectively used in the quoted statement. This forces us to conclude that this property is not effective in most fixed point results based on such contractive conditions. On the other hand, the argument used here is, practically, a clone of that developed for the standard metric setting. Hence – at least for such results – it cannot get us new insights for the considered matter. Clearly, the introduction of an additional order structure on X does not change this conclusion. Hence, the results in the area due to Altun and Erduran [3] are but formal copies of the ones (in standard metric spaces) due to Agarwal et al [2]. This is also true for the common fixed points question; when,

e.g., the results in Shobkolaei et al [19] are but a translation of the ones (in standard metric spaces) due to Jachymski [12]. Finally, we may ask whether this reduction scheme comprises as well the class of contraction maps in general complete partial metric spaces taken as in Ilić et al [10]. Formally, such results are not reducible to the above ones. But, from a technical perspective, this is possible; see Turinici [21] for details.

5. Triangular symmetrics

Let (X, d) be a symmetric space, taken as in (c01); and T be a selfmap of X.

Lemma 5. *Suppose that T is $(d, M; \varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, T is d-asymptotic.*

The argument is based on the evaluation (4.1); see also Zhu et al [23].

Now, by simply combining this with Theorem 1, we have (under (c01))

Theorem 4. *Suppose that T is $(d, M; \varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, T is a global Picard operator (modulo d).*

A basic particular case of this result is to be stated under the lines below. Call the symmetric $d(.,.)$, a *weak almost partial metric* provided it is triangular and sufficient (i.e.: (d02) holds). Note that, in such a case, relations (4.3) and (4.4) are still retainable. Assume in the following that

(e01) d is a weak almost partial metric and (X, d) is 0-complete.

Theorem 5. *Let the selfmap T be $(d, M; \varphi)$-contractive, for some right limit normal function $\varphi \in \mathcal{F}(R_+)$. Then, conclusions of Theorem 3 are holding.*

The proof mimics the one of Theorem 3 (if one takes Theorem 4 as starting point); so, it will be omitted.

Now, let us give two important examples of such objects.

(A) Clearly, each *(standard) metric* on X is a weak almost partial metric. Then, Theorem 5 is comparable with the main result in Jachymski [11].

(B) Remember that the symmetric d is called a *partial metric* provided it is reflexive-triangular and (d04)+(d05) hold. As before, (4.7) tells us (via (d04)) that each partial metric is a weak almost partial metric; hence, Theorem 5 is applicable to such objects. In particular, when φ is linear ($\varphi(t) = \lambda t$, $t \in R_+$, for some $\lambda \in [0, 1[$), one recovers the Banach type fixed point result in Aage and Salunke [1]; which, in turn, includes the one in Valero [22]. On the other hand, Lemma 3 tells us that Theorem 5 includes as well a related fixed point statement due to Dukić et al [6]; see also Golubović et al [8]; moreover, by Remark 1, the converse inclusion is not in general true. It is to be stressed here that the Matthews property (d05) was not effectively used in the quoted statement; in addition, the (stronger) reflexive-triangular property of d was replaced by the triangular property of the same. As before, the argument used here is, practically, a clone of that developed in the standard metric setting; whence, the results we just quoted are technically deductible from the one in Boyd and Wong [5]. Further developments of these results may be stated under the lines in Turinici [20].

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